

Path integrals in Snyder space

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Abstract

The definition of path integrals in one- and two-dimensional Snyder space is discussed in detail both in the traditional setting and in the first-order formalism of Faddeev and Jackiw.

1 Introduction

The interest in noncommutative spaces has increased in recent years, because they may describe the structure of space (or spacetime) at the Planck scale, as several approaches to quantum gravity seem to indicate [1]. The formulation of quantum mechanics on a noncommutative space is usually called noncommutative quantum mechanics. Path integral techniques have demonstrated to be convenient in the study of this theory.

A characteristic of noncommutative spaces is that the corresponding classical phase space is not canonical, i.e. the Poisson brackets do not have the usual form. However, the standard definition of path integral assumes a canonical phase space [2, 3], and one has therefore to extend the formalism to include this more general situation.

This is an interesting problem, that has been afforded in a variety of ways [4, 5]. However, most work on the subject has been developed for the so-called Moyal plane [6], a simple model whose Poisson brackets are constant tensors, hence necessarily implying the breakdown of the Lorentz invariance.

However, more general models of noncommutative spaces exist, in which the Lorentz invariance is preserved. The best known is the Snyder model [7], which, in spite of the presence in its definition of a parameter β with the dimension of inverse momentum, is Lorentz invariant. The quantum mechanics of the Snyder model has been studied in several papers [8, 9].

In its nonrelativistic version, the Snyder model is based on a deformation of the Heisenberg algebra, given by the commutation relations

$$[q_i, p_j] = i(\delta_{ij} + \beta^2 p_i p_j), \quad [q_i, q_j] = i\beta^2 J_{ij}, \quad [p_i, p_j] = 0, \quad (1)$$

where q_i and p_i are the phase space coordinates, and J_{ij} are the angular momentum generators; we use units in which $\hbar = 1$. Clearly, the classical limit of these commutators gives rise to a noncanonical phase space.

In this paper, we discuss the formulation of the nonrelativistic quantum mechanics of the Snyder model through path integral methods, following the approach of [4]. We also obtain the same results using the more formal techniques introduced in [10] for the study of first-order systems.

Recently, this problem has also been investigated in [11] in the case of one spatial dimension.

2 Noncanonical classical mechanics

Before discussing the path integral formulation of Snyder quantum mechanics, we shortly review some facts concerning noncanonical Hamiltonian formalism [12, 13], that will be useful in the following.

Let us consider noncanonical fundamental Poisson brackets

$$\{\xi_i, \xi_j\} = \Omega_{ij}(\xi), \quad (2)$$

where ξ_i denotes the phase space variables q_i and p_i and Ω_{ij} is an invertible matrix. Then the Hamilton equations for the Hamiltonian $H(\xi)$ read

$$\dot{\xi}_i = \Omega_{ij} \frac{\partial H}{\partial \xi_j}, \quad (3)$$

or equivalently,

$$(\Omega^{-1})^{ij} \dot{\xi}_j = \frac{\partial H}{\partial \xi_i}, \quad (4)$$

We want to obtain these equation from the variation of a first-order action of the form

$$I = \int [a^i(\xi) \dot{\xi}_i - H(\xi)] dt. \quad (5)$$

Then the condition

$$\frac{\partial a^j}{\partial \xi_i} - \frac{\partial a^i}{\partial \xi_j} = (\Omega^{-1})^{ij} \quad (6)$$

must hold. Solving (6) for the a^i , one can write down the action which generates the Hamilton equations (3).

3 One-dimensional Snyder path integral

Of course, when investigating the Snyder model, one must use the phase space formulation of the path integral. For a particle satisfying canonical Poisson brackets, moving in a one-dimensional space, this is given by

$$A = \int \mathcal{D}p \mathcal{D}q \, e^{iI}, \quad (7)$$

where

$$I = \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} (p\dot{q} - H(q, p)) dt \quad (8)$$

is the action (with L the Lagrangian and H the Hamiltonian), and $\mathcal{D}p\mathcal{D}q$ is a measure on the space of paths on phase space that will be defined below.

It can be shown that in a momentum basis the transition amplitude from an initial state of momentum p_i at time t_i to a final state of momentum p_f at time t_f is given by

$$\langle p_f | e^{-i\hat{H}(t_f-t_i)} | p_i \rangle = A. \quad (9)$$

We have chosen a momentum basis, because, when we shall consider Snyder space, the standard position variables will not commute and hence do not form a complete set of observables.

We wish to generalize this formula to the one-dimensional Snyder phase space (see also [11]), whose only nontrivial Poisson bracket is

$$\{q, p\} = 1 + \beta^2 p^2. \quad (10)$$

Given the Hamiltonian $H = \frac{p^2}{2} + V(q)$, the Hamilton equations in Snyder space read

$$\dot{q} = (1 + \beta^2 p^2)p, \quad \dot{p} = -(1 + \beta^2 p^2)\frac{\partial V}{\partial q}. \quad (11)$$

These equations can be obtained from an action principle, by modifying the definition of the Lagrangian so that (10) holds. Using the procedure of section 2, one gets the action

$$I = \int \left(-\frac{q\dot{p}}{1 + \beta^2 p^2} - H \right) dt = \int \left(\frac{\arctan \beta p}{\beta} \dot{q} - H \right) dt, \quad (12)$$

where the two expressions are equivalent modulo an integration by parts. We will now show that inserting (12) into (7) gives the correct expression for the path integral.

We first recall some results concerning the quantum mechanics of the one-dimensional Snyder model [14]. The Poisson bracket (10) goes into the commutator

$$[\hat{q}, \hat{p}] = i(1 + \beta^2 \hat{p}^2). \quad (13)$$

The operators \hat{q} and \hat{p} obeying (13) can be represented in a momentum basis by [15]

$$\hat{p} = p, \quad \hat{q} = i(1 + \beta^2 p^2)\frac{\partial}{\partial p}. \quad (14)$$

These operators are hermitian with respect to the scalar product

$$\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta^2 p^2} \psi^*(p) \phi(p). \quad (15)$$

The identity operator can therefore be expanded in terms of momentum eigenstates $|p\rangle$ as [14]

$$1 = \int_{-\infty}^{\infty} \frac{dp}{1 + \beta^2 p^2} |p\rangle \langle p|, \quad \text{with} \quad \langle p | p' \rangle = (1 + \beta^2 p^2) \delta(p - p'). \quad (16)$$

The eigenvalue equation for the position operator, $\hat{q} |q\rangle = q |q\rangle$, has formal solutions¹

$$\langle p | q \rangle \propto e^{-iq \frac{\arctan \beta p}{\beta}} \quad (17)$$

These states form an overcomplete set. However, one can choose a discrete basis with $q = 2\beta n$, n integer, which satisfies the completeness relation [14]

$$1 = \int dq |q\rangle \langle q|, \quad \text{with} \quad \langle q | q' \rangle = \delta_{qq'}. \quad (18)$$

For simplicity of notation, we have used an integral sign for the infinite sum over n .

Let us go back to the path integral. Splitting the interval $t_f - t_i$ into N intervals of length $\epsilon = t_k - t_{k-1}$, the standard definition reduces in our case to

$$A = \int \prod_{k=1}^{N-1} \frac{dp^{(k)}}{1 + \beta^2 p^{(k)2}} \int \prod_{k=1}^N dq^{(k)} \prod_{k=1}^N \langle p^{(k)} | q^{(k)} \rangle \langle q^{(k)} | e^{-i\epsilon \hat{H}} | p^{(k-1)} \rangle, \quad (19)$$

where the completeness relations (16) and (18) have been used. Recalling (17), one obtains

$$\langle q | e^{-i\epsilon \hat{H}} | p \rangle \propto e^{iq \frac{\arctan \beta p}{\beta} - i\epsilon H} \quad (20)$$

Finally, in the limit $\epsilon \rightarrow 0$, taking into account that $p_k - p_{k-1} \sim \dot{p} \epsilon \rightarrow \dot{p} dt$, one recovers for A the form (7) with

$$I = \int \left[-\frac{q\dot{p}}{1 + \beta^2 p^2} - H(q, p) \right] dt \quad (21)$$

¹These eigenstates are not physical, because they have infinite energy [14], but are sufficiently regular to adopt them in this setting.

and

$$\mathcal{D}p = \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \frac{dp^{(k)}}{1 + \beta^2 p^{(k)2}}, \quad \mathcal{D}q = \lim_{N \rightarrow \infty} \prod_{k=1}^N dq^{(k)}, \quad (22)$$

which proves our claim.

4 Two dimensional Snyder path integral

In higher dimensions the problem is more difficult, since the position operators \hat{q}_i do not commute and their eigenfunctions cannot be taken as a basis for the Hilbert space. It is convenient to use radial coordinates instead. In particular, we shall discuss the case $D = 2$, for which

$$\{q_i, p_j\} = \delta_{ij} + \beta^2 p_i p_j, \quad \{q_i, q_j\} = \beta^2 J_{ij}, \quad \{p_i, p_j\} = 0, \quad (23)$$

where $J_{ij} = p_j q_i - p_i q_j$. We choose polar coordinates to parametrize the momentum space, and their canonically conjugate variables for the position space. More precisely, we define

$$p_\rho = \sqrt{p_1^2 + p_2^2}, \quad p_\theta = \arctan \frac{p_2}{p_1}, \quad (24)$$

and

$$\rho = \frac{p_1 q_1 + p_2 q_2}{\sqrt{p_1^2 + p_2^2}}, \quad J = J_{12} = p_2 q_1 - p_1 q_2. \quad (25)$$

The Jacobian of the transformation is 1. Note that classically $r^2 \equiv q_1^2 + q_2^2 = \rho^2 + J^2/p_\rho^2$.

The position coordinates ρ and J essentially correspond to the parallel and orthogonal components of the position vector with respect to the momentum of the particle. The phase space polar coordinates defined above obey the Poisson brackets

$$\begin{aligned} \{p_\rho, p_\theta\} &= 0, & \{\rho, J\} &= 0, & \{p_\rho, J\} &= 0, \\ \{p_\rho, \rho\} &= 1 + \beta^2 p_\rho^2, & \{p_\theta, J\} &= 1, & \{p_\theta, \rho\} &= 0. \end{aligned} \quad (26)$$

Notice that the Poisson bracket of ρ and J vanishes, and hence the corresponding quantum operators commute and form a basis for the position space. An alternative basis would be constituted by the operators \hat{r}^2 and \hat{J} , where $\hat{r}^2 \equiv \hat{q}_1^2 + \hat{q}_2^2$ [9]. These coordinates however give rise to more complicated formulas.

Again, by the methods of section 2, one can obtain the action that generates the classical Hamilton equations through the Poisson brackets (26). This is

$$I = - \int \left[\frac{\rho \dot{p}_\rho}{1 + \beta^2 p_\rho^2} + J \dot{p}_\theta + H \right] dt = \int \left[\frac{\arctan \beta p_\rho}{\beta} \dot{\rho} + p_\theta \dot{J} - H \right] dt \quad (27)$$

where

$$H = \frac{p_\rho^2}{2} + V(\rho, J). \quad (28)$$

Let us now consider the quantum theory. The quantum operators must be defined carefully, because of ordering ambiguities. We adopt an ordering with \hat{p}_i always on the left. The commutation relations are

$$[\hat{q}_i, \hat{p}_j] = i(\delta_{ij} + \beta^2 \hat{p}_i \hat{p}_j), \quad [\hat{q}_i, \hat{q}_j] = i\hat{J}_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0 \quad (29)$$

A representation of (29) is given by [15]

$$\hat{p}_i = p_i, \quad \hat{q}_i = i \left(\frac{\partial}{\partial p_i} + \beta^2 p_i p_j \frac{\partial}{\partial p_j} \right) \quad (30)$$

The hermitian operators corresponding to the classical polar coordinates are

$$\hat{p}_\rho = \sqrt{p_i^2} \equiv p_\rho, \quad \hat{p}_\theta = \arctan \frac{p_2}{p_1} \equiv p_\theta, \quad (31)$$

and

$$\hat{\rho} = i(1 + \beta^2 p_\rho^2) \left(\frac{\partial}{\partial p_\rho} + \frac{1}{2p_\rho} \right), \quad \hat{J} = i \frac{\partial}{\partial p_\theta}, \quad (32)$$

where the scalar product

$$\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{p_\rho dp_\rho dp_\theta}{1 + \beta^2 p_\rho^2} \psi^*(p) \phi(p) \quad (33)$$

is understood. The completeness relations for momentum eigenstates are therefore

$$\int_{-\infty}^{+\infty} \frac{p_\rho dp_\rho}{1 + \beta^2 p_\rho^2} \int_0^{2\pi} dp_\theta |p_\rho, p_\theta\rangle \langle p_\rho, p_\theta| = 1 \quad (34)$$

The eigenvalue equations for the position operators read²

$$\hat{\rho} |\rho\rangle = \rho |\rho\rangle, \quad |\rho\rangle \propto \frac{e^{-i\rho \frac{\arctan \beta p_\rho}{\beta}}}{\sqrt{p_\rho}}, \quad (35)$$

²Also in this case the eigenfunction are not physical because their energy diverges.

and

$$\hat{J} |J\rangle = J |J\rangle, \quad |J\rangle \propto e^{-iJp_\theta}, \quad (36)$$

with integer J .

Defining a basis $|\rho, J\rangle = |\rho\rangle |J\rangle$, one has

$$\langle p_\rho, p_\theta | \rho, J \rangle = \frac{e^{-i(\rho \frac{\arctan \beta p_\rho}{\beta} + Jp_\theta)}}{\sqrt{p_\rho}}. \quad (37)$$

with

$$\int_0^\infty d\rho \int_{-\infty}^\infty dJ |\rho, J\rangle \langle \rho, J| = 1. \quad (38)$$

Hence,

$$\langle \rho, J | e^{-i\epsilon H} | p_\rho, p_\theta \rangle = \frac{e^{i(\rho \frac{\arctan \beta p_\rho}{\beta} + Jp_\theta - \epsilon H)}}{\sqrt{p_\rho}}. \quad (39)$$

From (39), proceeding as in the one-dimensional case one obtains in the limit $\epsilon \rightarrow 0$ the formula (7) with action (27), and measure adapted to two dimensions,

$$\mathcal{D}p = \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int \frac{d^2 p^{(k)}}{1 + \beta^2 p^{(k)2}}, \quad \mathcal{D}q = \lim_{N \rightarrow \infty} \prod_{k=1}^N \int d^2 q^{(k)}, \quad (40)$$

where $d^2 p^{(k)} = p_\rho^{(k)} dp_\rho^{(k)} dp_\theta^{(k)}$ and $d^2 q^{(k)} = d\rho^{(k)} dJ^{(k)} / p_\rho^{(k)}$.

Moreover, in terms of the previous coordinates, the classical Hamiltonian $H = \frac{p_\rho^2}{2} + V(r^2)$ takes the form $H = \frac{p_\rho^2}{2} + V\left(\rho^2 + \frac{J^2}{p_\rho^2}\right)$. However, it is known that in order to obtain the correct result from the path integral, that takes account of the hermitian nature of the operator $\hat{\rho}^2$, an additional term $-1/2p_\rho^2$ must be added to the classical two-dimensional action [16]. Hence, the correct effective potential will be $V = V\left(\rho^2 + \frac{J^2 - 1/2}{p_\rho^2}\right)$.

5 Faddeev-Jackiw formalism

It is remarkable that the previous results can be obtained in an easier way using the first-order formalism introduced by Faddeev and Jackiw. In fact, it is shown in [10, 13], using a Darboux transformation from the original

variables ξ_i to new canonical variables, that the path integral can be written as

$$A = \int \mathcal{D}\xi_i |\det \Omega_{ij}|^{-1/2} e^{iI}, \quad (41)$$

where the determinant arises from the Jacobian of the transformation and I is given by (5).

In our case, in any dimension,

$$|\det \Omega_{ij}| = (1 + \beta^2 p_i^2)^2, \quad (42)$$

from which the measures (22) and (40) follow.

Notice that this method can be easily employed to study different non-commutative models, including for example the Moyal plane, investigated in [4] by different means.

6 Two-dimensional examples

In this section, we give a few elementary examples of application of the formalism to two-dimensional models.

In the case of a free particle, the integration over the angular variables p_θ and J simply yields a delta function $\delta(p_\theta^{(i)} - p_\theta^{(f)})$, and one is left with an integral over the radial coordinates,

$$\int \mathcal{D}\rho \int \mathcal{D}p_\rho \exp \left[i \int \left(\frac{\rho \dot{p}_\rho}{1 + \beta^2 p_\rho^2} + \frac{p_\rho^2}{2} \right) dt \right]. \quad (43)$$

where

$$\mathcal{D}p_\rho = \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int \frac{dp_\rho^{(k)}}{1 + \beta^2 p_\rho^{(k)2}}. \quad (44)$$

Performing a change of variables $P_\rho = \beta^{-1} \arctan \beta p_\rho$, one gets

$$\int \mathcal{D}\rho \int \mathcal{D}P_\rho \exp \left[i \int \left(\rho \dot{P}_\rho + \frac{\tan^2 P_\rho}{2} \right) dt \right], \quad (45)$$

with

$$\mathcal{D}P_\rho = \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int dP_\rho^{(k)}. \quad (46)$$

The integration on ρ gives in turn gives a delta function $\delta(P_\rho^{(i)} - P_\rho^{(f)})$ and one is left with

$$\int \mathcal{D}P_\rho \exp \left[\frac{i}{2} \int \tan^2 P_\rho dt \right]. \quad (47)$$

In the harmonic oscillator case, the classical potential is $V = \omega^2 r^2 \rightarrow \omega^2 \left(\rho^2 + \frac{J^2 - 1/4}{p_\rho^2} \right)$. It is now convenient to integrate first in p_θ , getting the conservation of the angular momentum, $\delta(J^{(i)} - J^{(f)})$. The integral reduces then to a sum on different J sectors:

$$\int \mathcal{D}\rho \int \frac{\mathcal{D}p_\rho}{1 + \beta^2 p_\rho^2} \exp \left[i \int \left(\frac{\rho \dot{p}_\rho}{1 + \beta^2 p_\rho^2} + \frac{p_\rho^2}{2} + \omega^2 \rho^2 + \omega^2 \frac{J^2 - 1/4}{p_\rho^2} \right) dt \right]. \quad (48)$$

Again defining a new variable P_ρ as before, one gets

$$\int \mathcal{D}\rho \int \mathcal{D}P_\rho \exp \left[i \int \left(\rho \dot{P}_\rho + \omega^2 \rho^2 + \frac{\tan^2 P_\rho}{2} + \omega^2 \frac{J^2 - 1/4}{\tan^2 P_\rho} \right) dt \right], \quad (49)$$

and the gaussian integration over ρ yields

$$\int \mathcal{D}P_\rho \exp \left[-i \int \left(\frac{\dot{P}_\rho^2}{4\omega^2} - \frac{\tan^2 P_\rho}{2} - \omega^2 \frac{J^2 - 1/4}{\tan^2 P_\rho} \right) dt \right]. \quad (50)$$

This path integral can be evaluated at least perturbatively by standard methods, and is similar to that obtained in the one-dimensional case in [11].

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